Binary Relations II

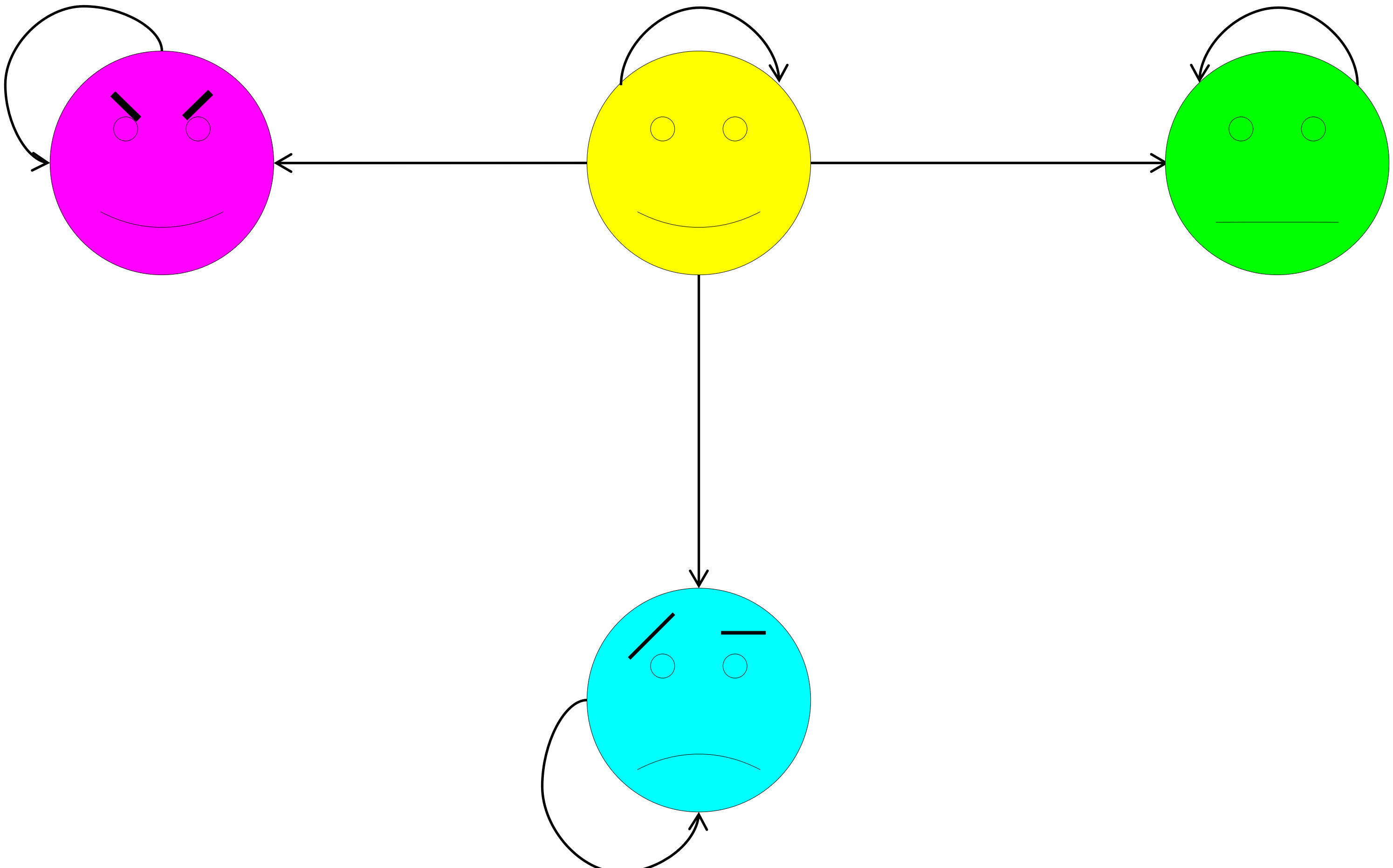
Outline for Today

Proofs Involving Binary Relations

- Equivalence Relation Proofs
- An Alternate Perspective on Partitions
- Proofs Involving Multiple Relations

Recap from Last Time

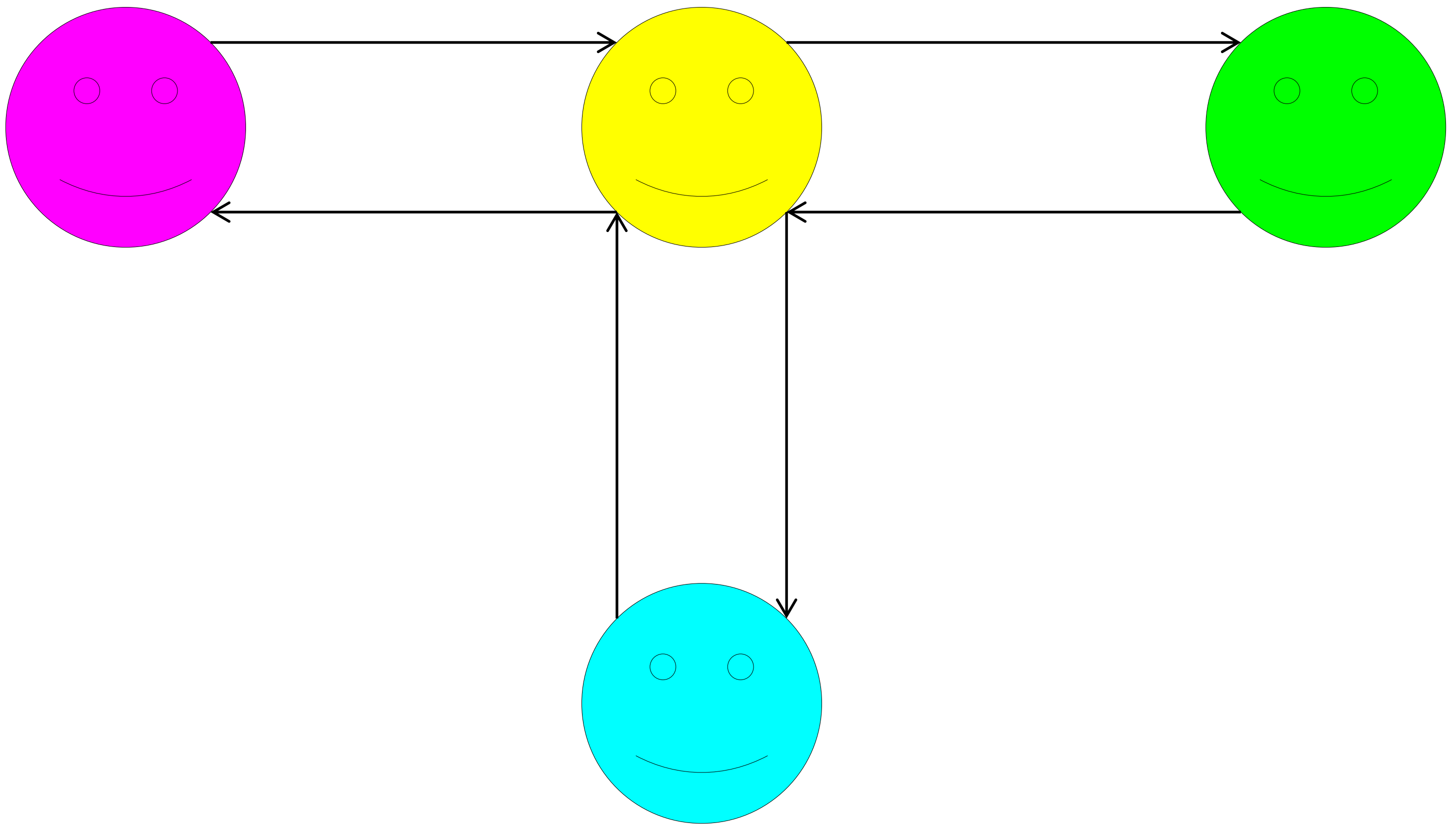
Reflexivity



$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)

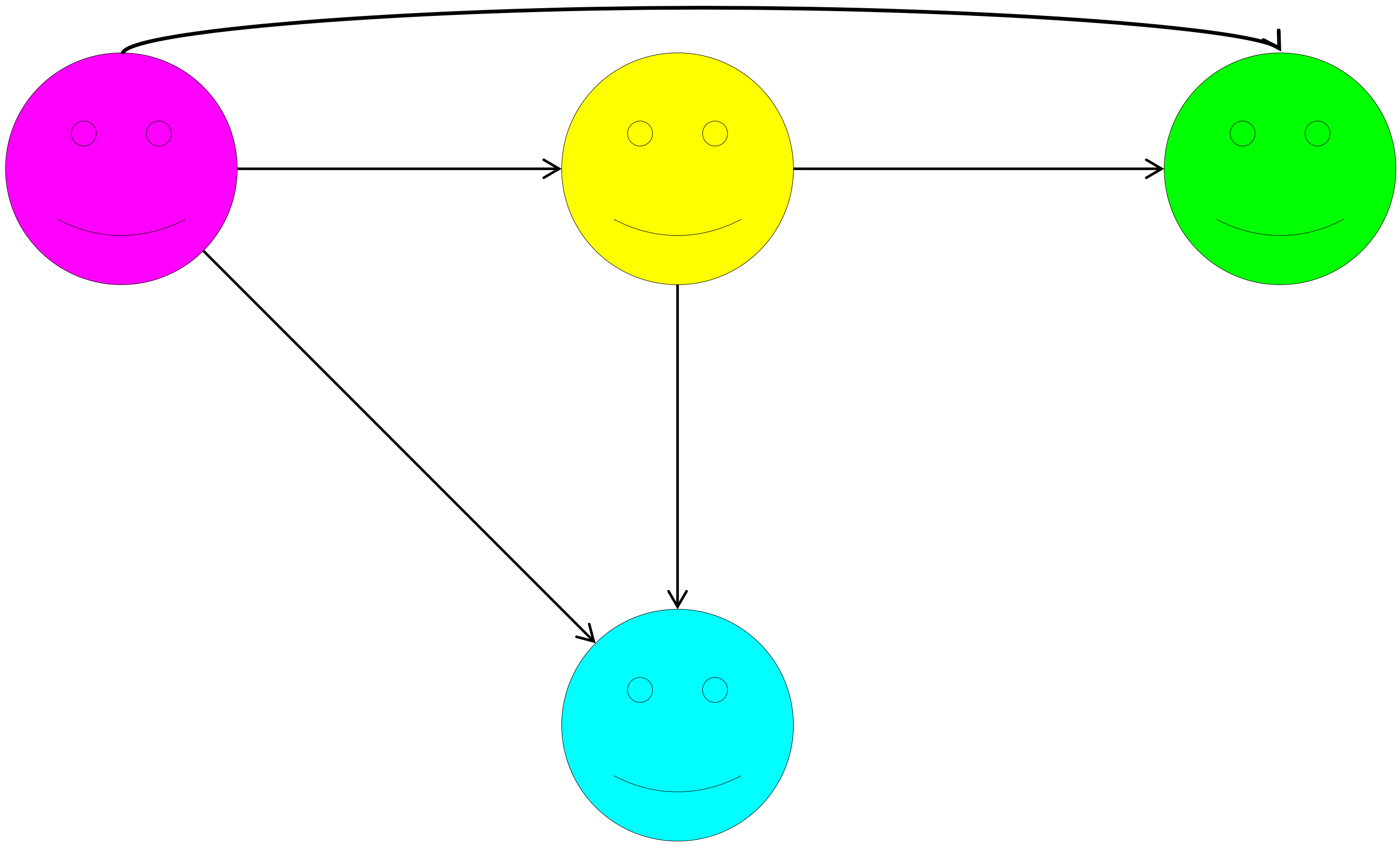
Symmetry



$$\forall a \in A. \forall b \in A. (aRb \rightarrow bRa)$$

(*"If a is related to b, then b is related to a."*)

Transitivity



$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

(*"Whenever a is related to b and b is related to c, we know a is related to c."*)

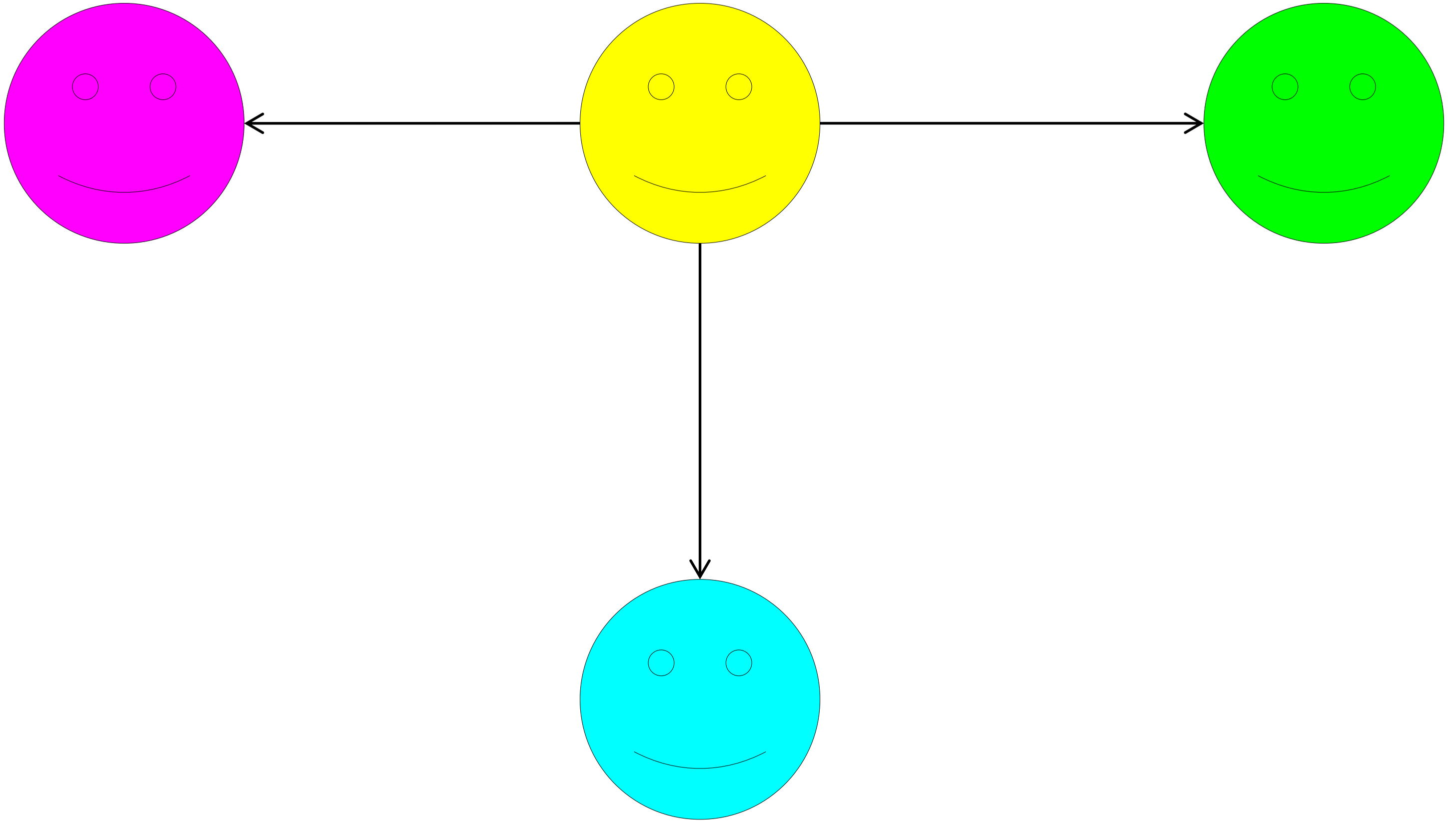
Equivalence Relations

An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.

Some examples:

- $x = y$
- $x \equiv_k y$
- x has the same color as y
- x has the same shape as y.

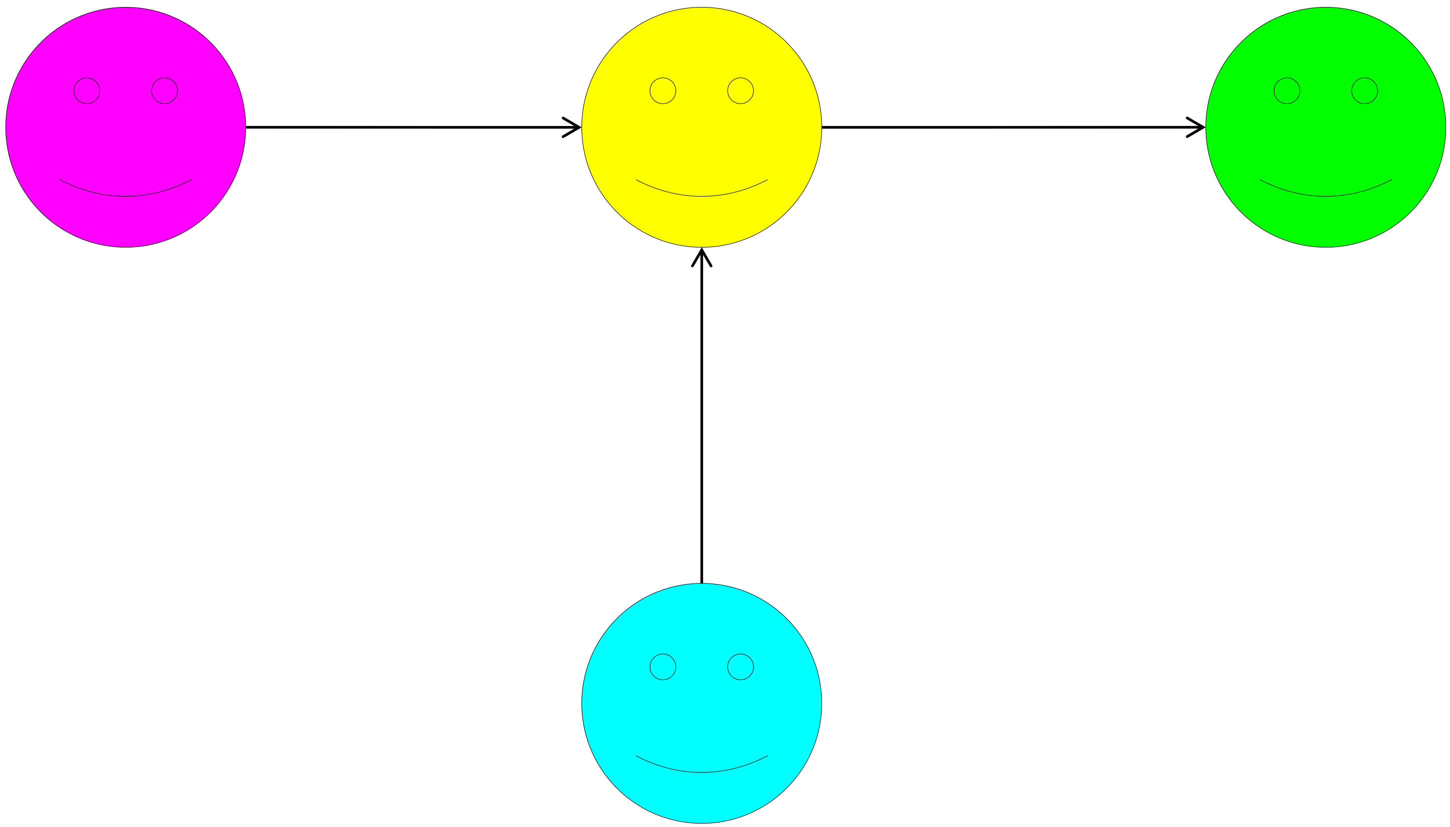
Irreflexivity



$$\forall a \in A.\ a \not R a$$

(*"No element is related to itself."*)

Asymmetry



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

(*"If a relates to b, then b does not relate to a."*)

Strict Orders

A ***strict order*** is a relation that is irreflexive, asymmetric and transitive.

Some examples:

$$x < y.$$

$$a \text{ can run faster than } b.$$

$$A \subsetneq B \text{ (that is, } A \subseteq B \text{ and } A \neq B\text{).}$$

Let’s do some proofs!

Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

An Example Relation

Consider the binary relation ~ defined over the set ℤ:

a~b **if** ***a+b* is even**

Some examples:

0~4 1~9 2~6 5~5

Turns out, this is an equivalence relation! Let's see how to prove it.

We can define binary relations by giving a rule, like this:

a~b **if** ***some property of a and b holds***

This is the general template for defining a relation. Although we're using “if” rather than “iff” here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with “if” rather than “iff.” Check the “Mathematical Vocabulary” handout for details.

What properties must ~ have to be an equivalence relation?

Reflexivity

Symmetry

Transitivity

Let's prove each property independently.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

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Lemma 1: The binary relation ~ is reflexive.

Proof:

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Therefore, we'll choose an arbitrary integer a, then go prove that $a \sim a$.

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***a*~*b* if *a*+*b* is even**

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***$a \sim b$* if *$a+b$* is even**

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To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even.

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Lemma 2: The binary relation $\sim$ is symmetric.

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Lemma 3: The binary relation $\sim$ is transitive.

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So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required.

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An Observation

$a \sim b$ if $a+b$ is even

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The formal definition of reflexivity is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

$a \sim b$ if $a+b$ is even

Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

Since $a \sim b$, we know that $a+b$ is even. Because $a+b = b+a$, this means that $b+a$ is even. Since $b+a$ is even, we know that $b \sim a$, as required. ■

The formal definition of symmetry is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

a~*b* if *a*+*b* is even

Lemma 3: The binary relation ~ is transitive.

Proof: Consider arbitrary integers *a*, *b* and *c* where $a \sim b$ and $b \sim c$. We need to prove that $a \sim c$, meaning that we need to show that $a+c$ is even.

Since $a \sim b$ and $b \sim c$, we know that $a+b$ and $b+c$ are even. This means there are integers *k* and *m* where $a+b = 2k$ and $b+c = 2m$. Notice that

$$(a+b) + (b+c) = 2k + 2m.$$

Rearranging, we see that

$$a+c + 2b = 2k + 2m,$$

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$$a+c = 2k + 2r$$

So there is an integer *r*, name

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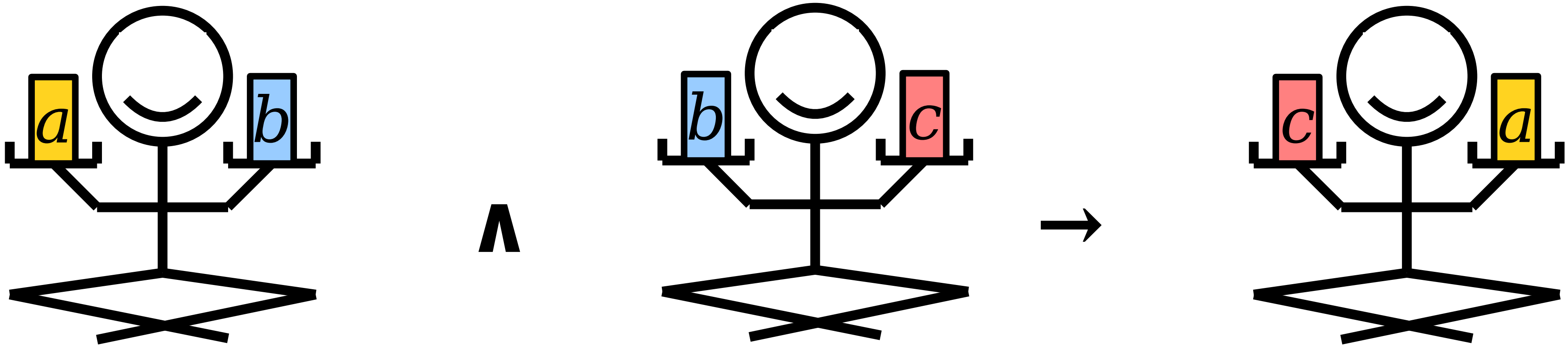
First-Order Logic and Proofs

- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
- Use the FOL definitions to determine what to assume and what to prove.
- Write the proof in plain English using the conventions we set up in the first week of the class.
- ***Please, please, please, please, please internalize the contents of this slide!***

Another Perspective on Partitions

The question we are asking the sage: “Are these two in the same equivalence class?”



$$aRb \quad \wedge \quad bRc \quad \rightarrow \quad cRa$$

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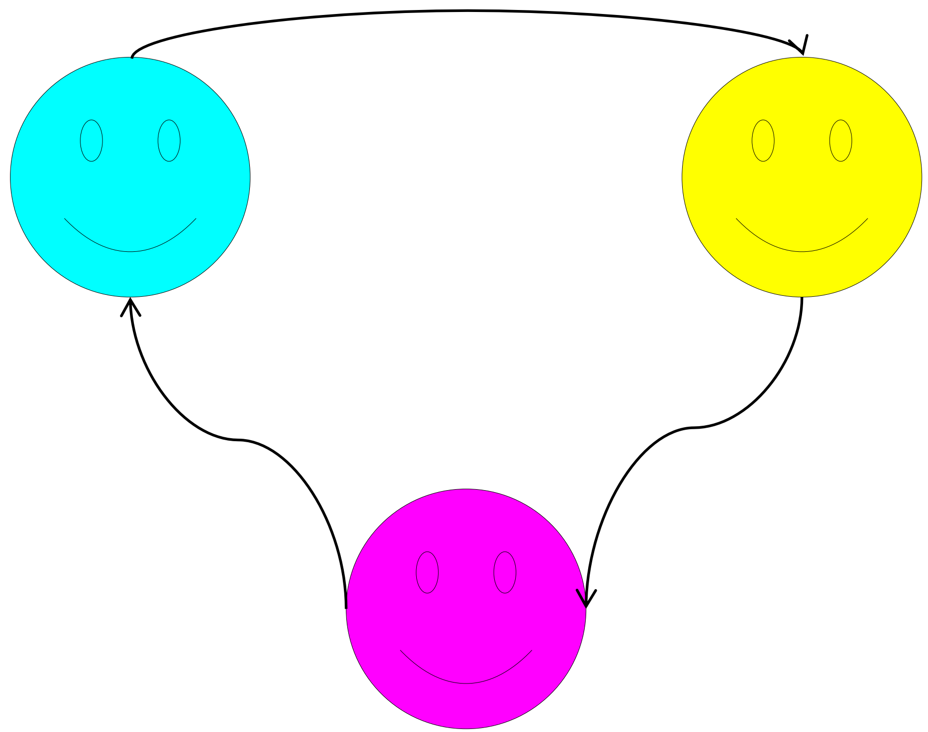

A binary relation
with this property
is called ***cyclic***.

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

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What We're Assuming

- R is an equivalence relation.
- R is reflexive.
- R is symmetric.
- R is transitive.

What We Need To Show

- R is reflexive.
- R is cyclic.

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R is reflexive.

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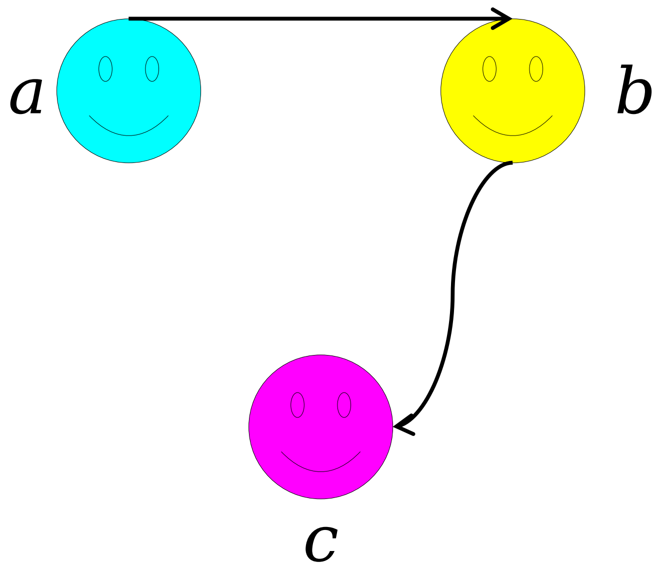

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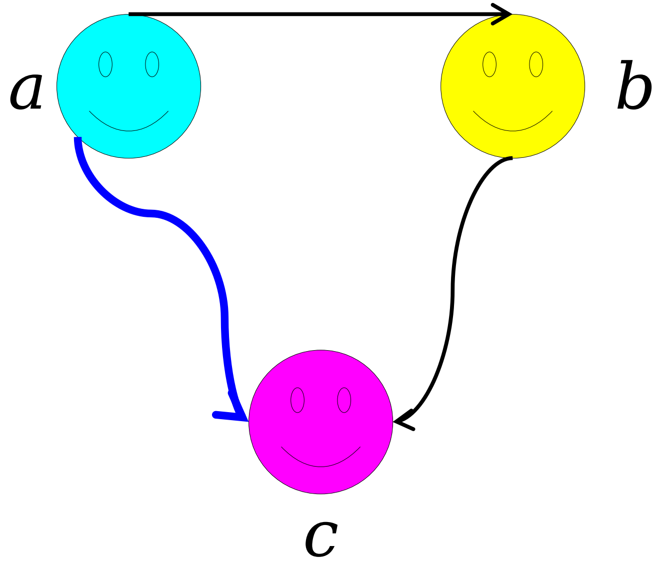

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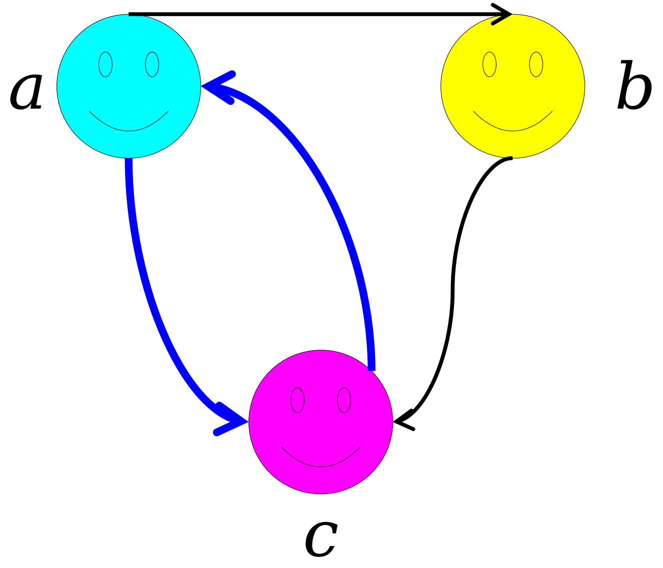

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Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

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Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc.

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To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds.

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To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc.

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To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove.

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Proof: Let R be an arbitrary equivalence relation over some set A. We need to prove that R is reflexive and cyclic.

Since R is an equivalence relation, we know that R is reflexive, symmetric, and transitive. Consequently, we already know that R is reflexive, so we only need to show that R is cyclic.

To prove that R is cyclic, consider any arbitrary $a, b, c \in A$ where aRb and bRc. We need to prove that cRa holds. Since R is transitive, from aRb and bRc we see that aRc. Then, since R is symmetric, from aRc we see that cRa, which is what we needed to prove. ■

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Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We're just following the templates from the first week of class!

Lemma 1: If R is an equivalence relation over a set A, then R is re…

Proof: Let … he set A … .

Notice how this setup mirrors the first-order definition of cyclicity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

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Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.

Lemm … n R is reflexive and cyclic.

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- R is reflexive.
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What We Need To Show

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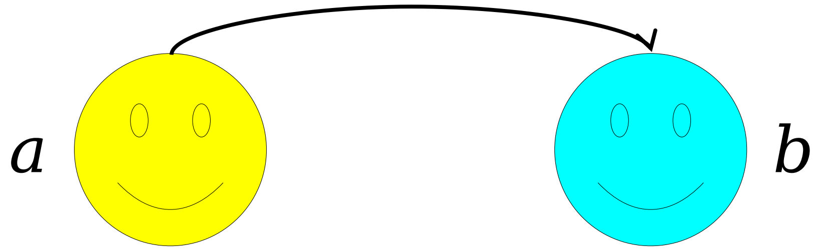

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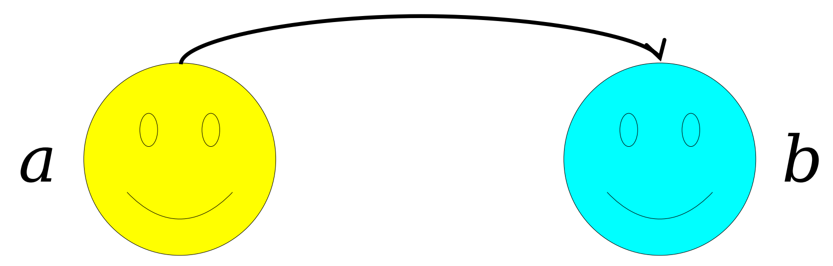

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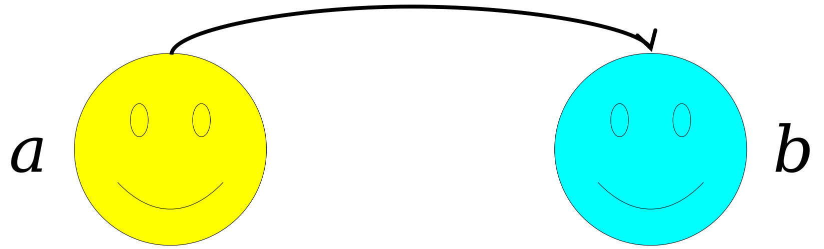

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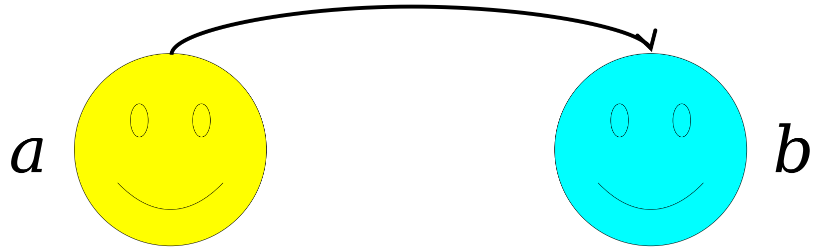

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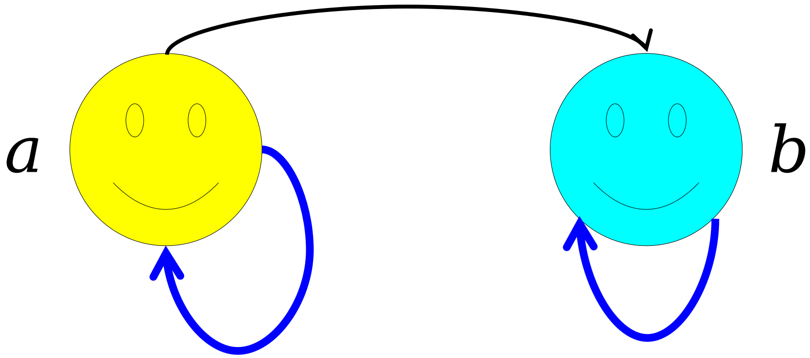

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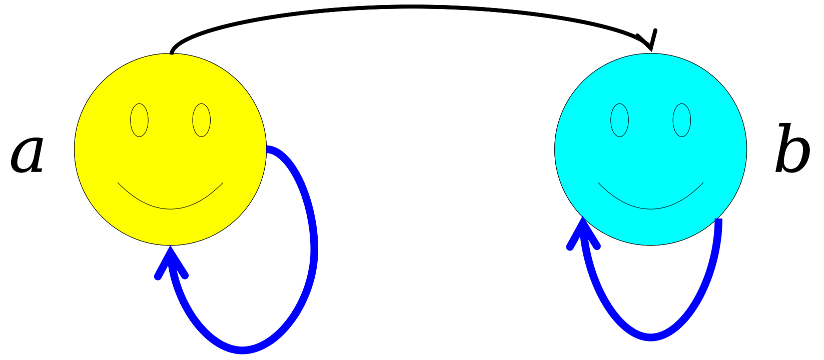

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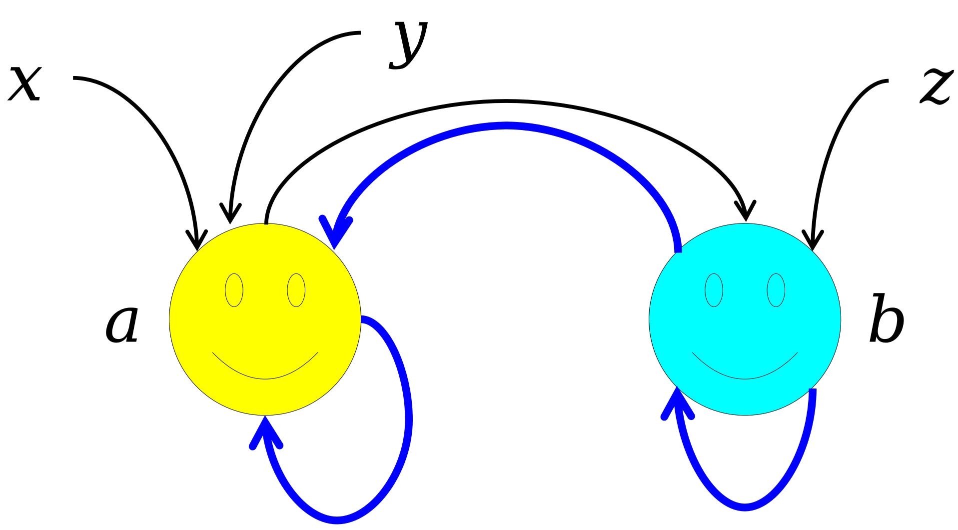

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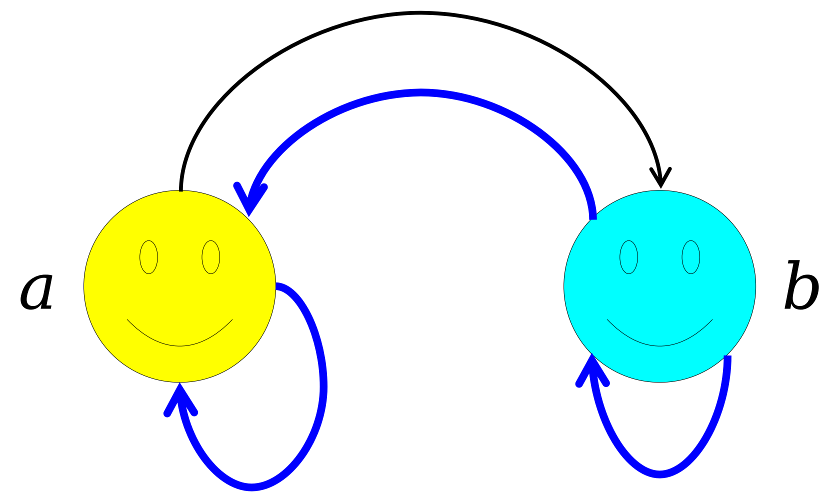

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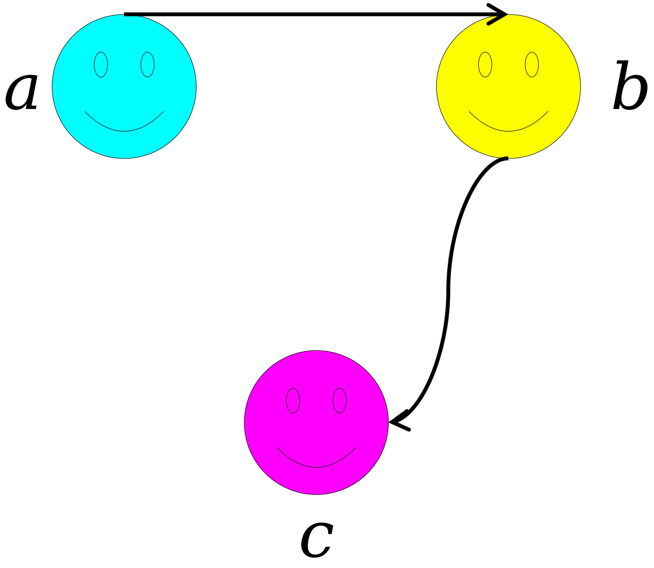

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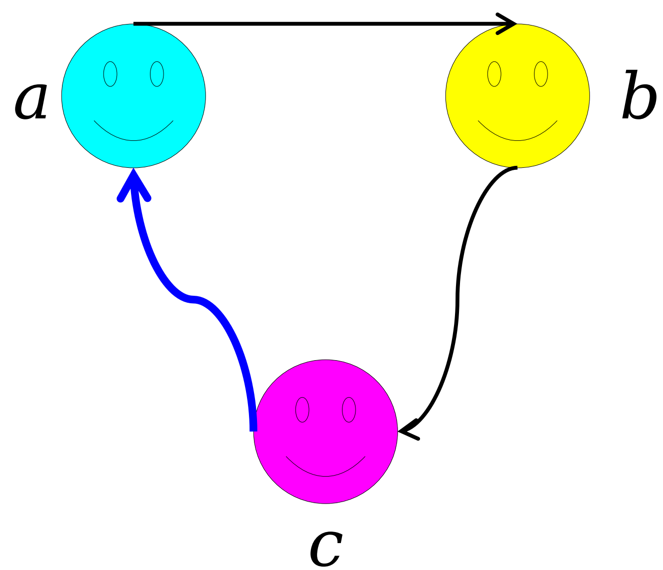

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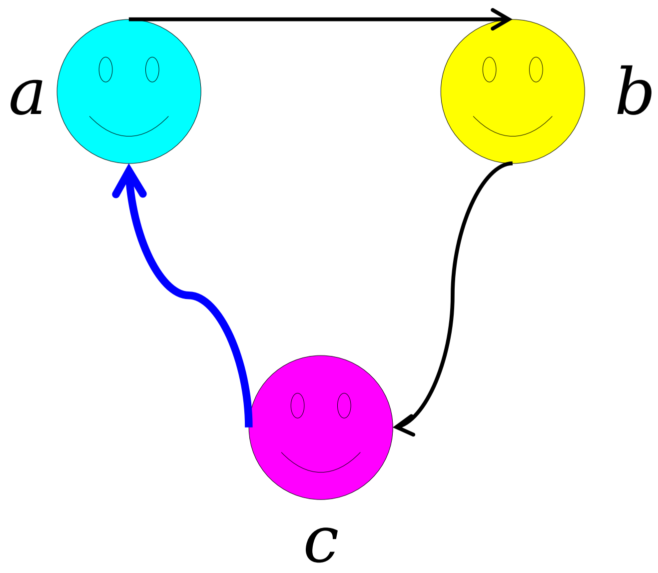

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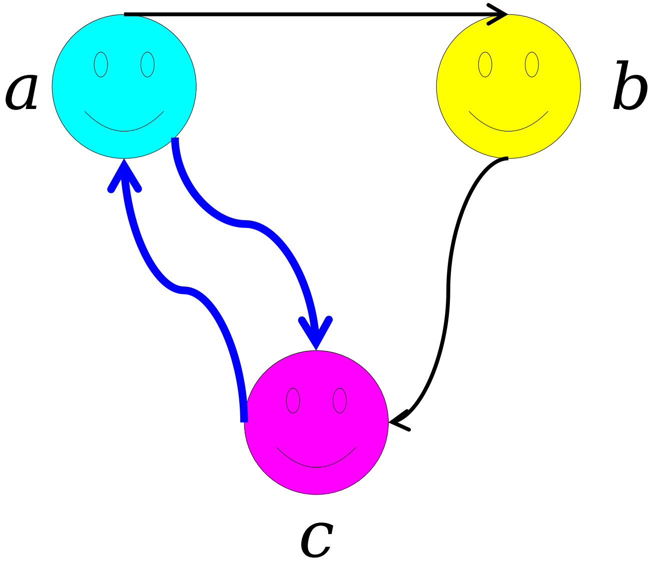

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Notice how this setup mirrors the first-order definition of symmetry:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

When writing proofs about terms with first-order definitions, it's critical to call back to those definitions!

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Refining Your Proofwriting

- When writing proofs about terms with formal definitions, you ***must*** call back to those definitions.
- Use the first-order definition to see what you'll assume and what you'll need to prove.
- When writing proofs about terms with formal definitions, you ***must not*** include any first-order logic in your proofs.
- Although you won't use any FOL *notation* in your proofs, your proof implicitly calls back to the FOL definitions.
- You'll get a lot of practice with this on Problem Set Three. If you have any questions about how to do this properly, please feel free to ask on Piazza or stop by office hours!

Let’s take a five minute break!

Communication Methods

Campuswire Posts

>

Staff Email

>

Personal Email

Proofs Involving Multiple Relations

Let R be a binary relation over a set A. We can define a new relation over A called the ***inverse relation of R***, denoted R^{-1}, as follows:

$$xR^{-1}y \qquad \text{if} \qquad yRx$$

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

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Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Before we attempt the prove/disprove, when it's a good idea to ***apply new definitions to a concrete example*** and make sure we fully understand what the definition means.

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Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

What is the inverse of the < relation over $\mathbb{Z}$?

What is the inverse of the = relation over $\mathbb{Z}$?

Discuss with your neighbors!

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What is the inverse of the $<$ relation over $\mathbb{Z}$?

The inverse of the $<$ relation over $\mathbb{Z}$ is the $>$ relation over $\mathbb{Z}$. This is because $x < y$ happens precisely when $y > x$.

What is the inverse of the $=$ relation over $\mathbb{Z}$?

The $=$ relation over $\mathbb{Z}$ is its own inverse. Note that $x = y$ happens precisely when $y = x$ happens.

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A good strategy for "prove or disprove" questions is to just try doing both a proof and a disproof.

If you find yourself having a hard time proving the claim, identifying *why* can often help you come up with a disproof and vice versa.

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How would you set up a proof of this claim?

How would you set up a disproof of this claim?

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Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

How would you set up a proof of this claim?

For an arbitrary relation R, assume that R is an equivalence relation, then show that R^{-1} also has to be an equivalence relation.

How would you set up a disproof of this claim?

Find a specific example of a relation R such that R is an equivalence relation but R^{-1} is not.

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

What We're Assuming

R is an equivalence relation

What We Need To Show

R^{-1} is an equivalence relation

Relevant Definitions

$$xR^{-1}y \text{ if } yRx$$

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

What We're Assuming

R is an equivalence relation

- **R is reflexive**
- **R is symmetric**
- **R is transitive**

What We Need To Show

R^{-1} is an equivalence relation

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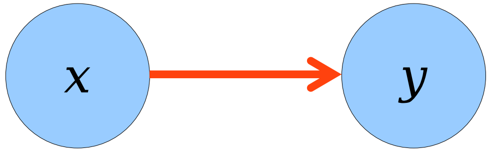
Relevant Definitions

$$xR^{-1}y \text{ if } yRx$$

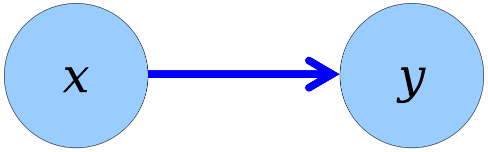
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

A great proofwriting strategy is to ***draw pictures*** – it's often easier to reason about concrete circles, lines, and arrows than abstract mathematical definitions.

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.



We'll use a ***red arrow*** to denote that x$\boldsymbol{R}y$



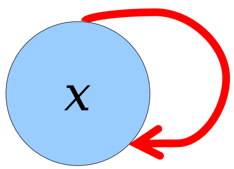
And a ***blue arrow*** to denote that x$\boldsymbol{R^{-1}}y$

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Assumptions:

R is reflexive

$\forall x \in A.\ x\mathbf{R}x$



We can always draw a red self-loop

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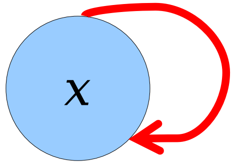

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If there's a red arrow in one direction, we can draw one in the other direction

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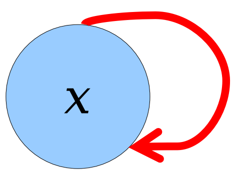

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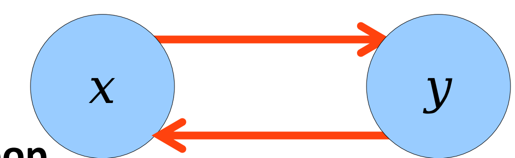
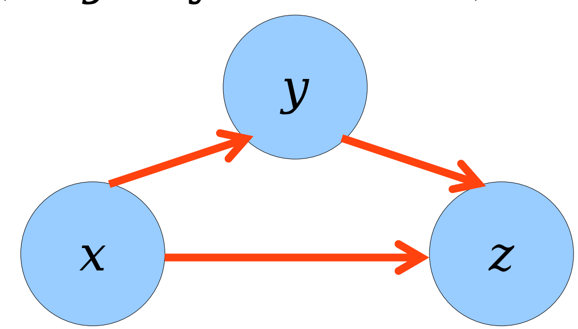


If there's a red arrow in one direction, we can draw one in the other direction

R is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ (x\mathbf{R}y \land y\mathbf{R}z \rightarrow x\mathbf{R}z)$

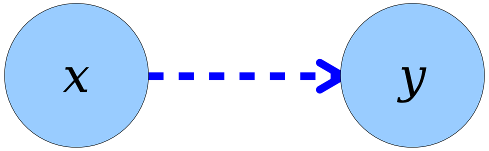


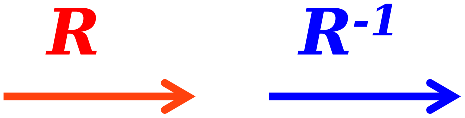
If you can get somewhere by following red arrows, you can draw a red arrow directly there

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

$$xR^{-1}y \text{ if } yRx$$

When can we draw a blue arrow?



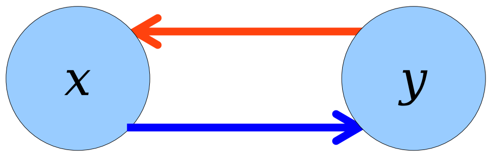


Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

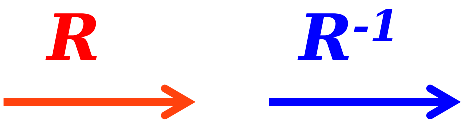
$xR^{-1}y$ if yRx

When can we draw a blue arrow?

If there's a red arrow going one way



Then we can draw a blue arrow going the other way



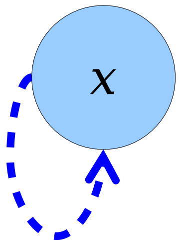
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

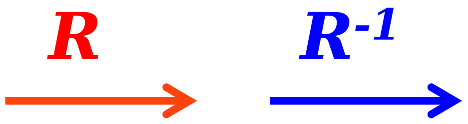
Goal:

R^{-1} is reflexive

$\forall x \in A.\ x R^{-1} x$

We want to always be able to draw a blue self-loop





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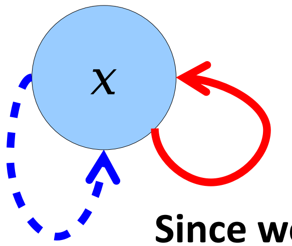
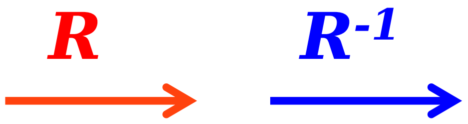


Since we assumed R is reflexive, we can put in this red self loop



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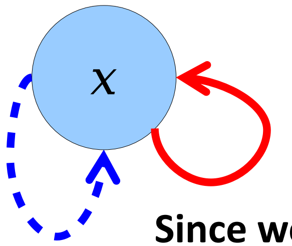
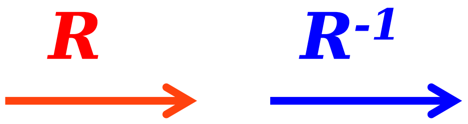
R^{-1} is reflexive

$\forall x \in A.\ x R^{-1} x$



Since we assumed R is reflexive, we can put in this red self loop

Since there's a red arrow going from x to x, we can draw a blue arrow going "the other way", from x to x



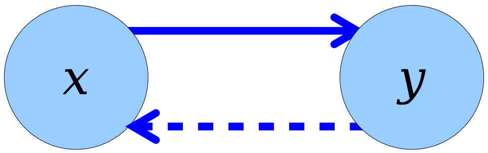
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

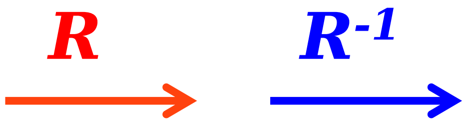
Goal:

R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \rightarrow y R^{-1} x)$

We want to say that if there's a blue arrow in one direction, we can draw one in the other direction





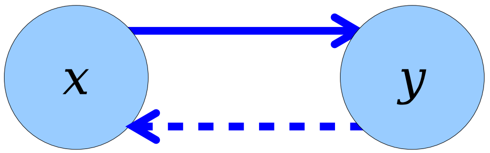
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Goal:

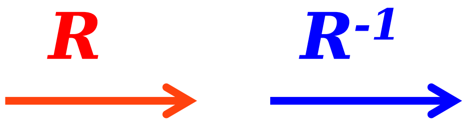
$\boldsymbol{R^{-1}}$ is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \rightarrow y R^{-1} x)$

So we'll assume this arrow exists



And prove that this arrow exists too



Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

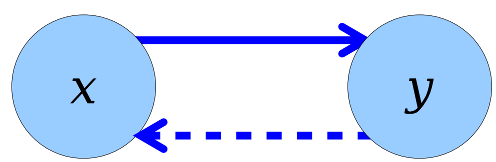
Goal:

R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \rightarrow y R^{-1} x)$

Talk with your neighbors and see if you can work out how to do this.

So we'll assume this arrow exists

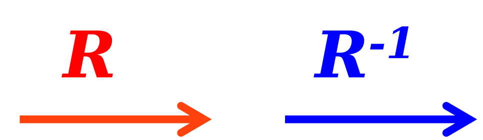


And prove that this arrow exists too

Remember that you can apply this definition

$x R^{-1} y$ if $y R x$

in the other direction too



Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

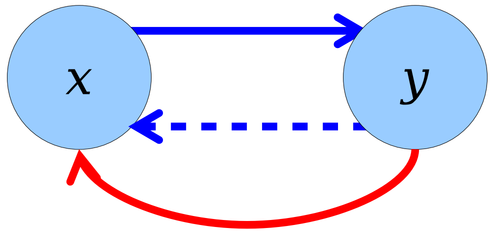
Goal:

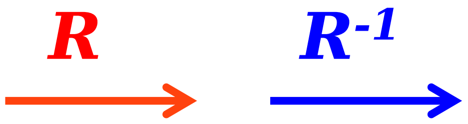
R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \rightarrow y R^{-1} x)$

$x R^{-1} y$ if $y R x$

Since there's a blue arrow from x to y, we can draw a red arrow going the other way, from y to x





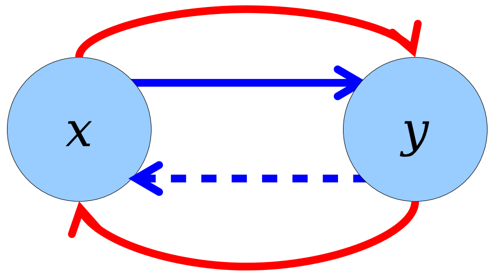
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

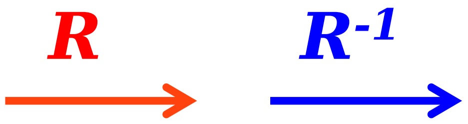
Goal:

R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(xR^{-1}y \rightarrow yR^{-1}x)$

Since R is symmetric, we can use this arrow to draw a red arrow from x to y



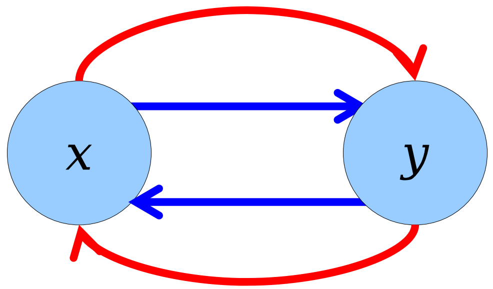


Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Goal:

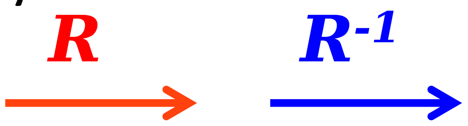
R^{-1} is symmetric

$\forall x \in A.\ \forall y \in A.$
$(x R^{-1} y \rightarrow y R^{-1} x)$



$x R^{-1} y$ if $y R x$

Finally, since we have a red arrow from x to y,
we can apply the definition of R^{-1} again to conclude that there's a blue arrow from y to x



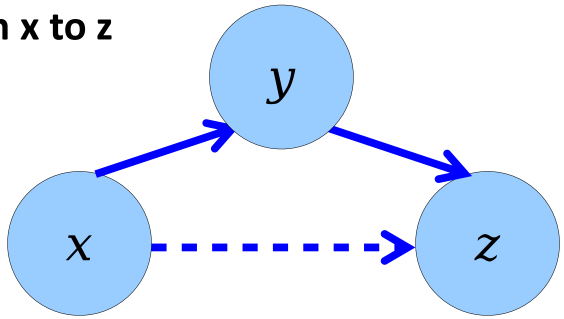
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

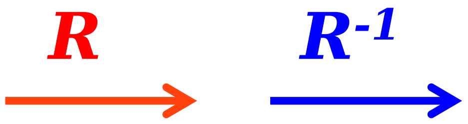
Goal:

R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(xR^{-1}y \land yR^{-1}z \to xR^{-1}z)$

We want to say that if we can get from x to z through an intermediary y, then we can draw an arrow straight from x to z





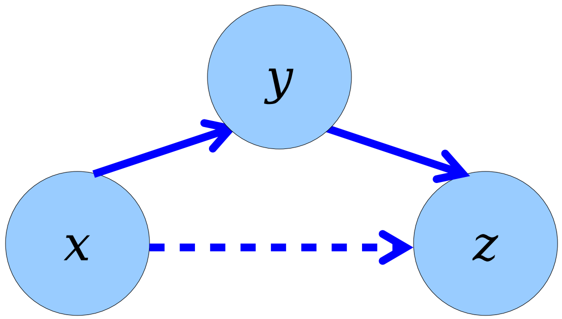
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Goal:

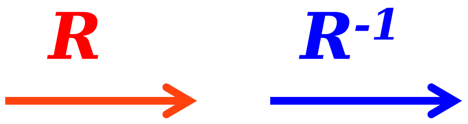
R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(xR^{-1}y \land yR^{-1}z \to xR^{-1}z)$

So we'll assume that these arrows exist



And prove that this arrow exists too



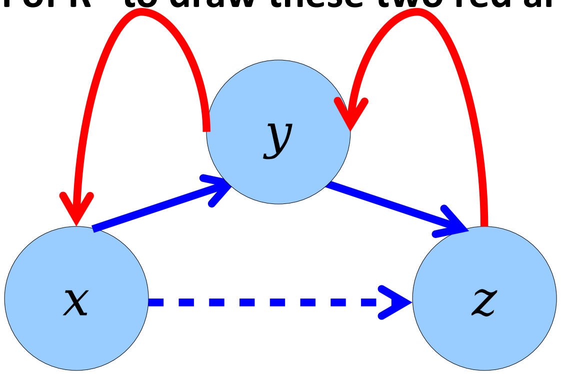
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

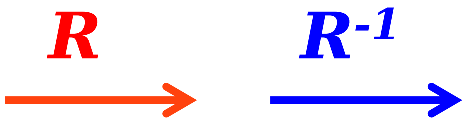
Goal:

R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(xR^{-1}y \land yR^{-1}z \rightarrow xR^{-1}z)$

We can apply the definition of R^{-1} to draw these two red arrows



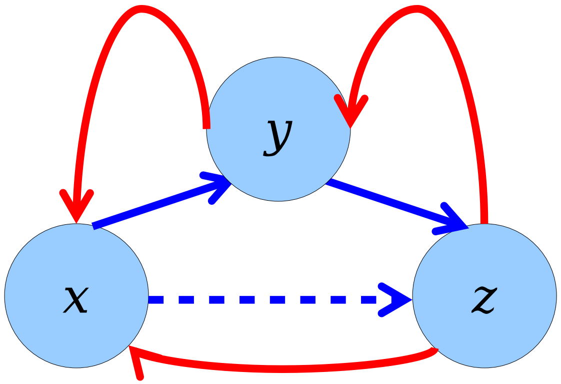


Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

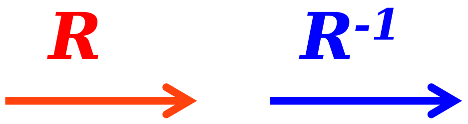
Goal:

R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(xR^{-1}y \wedge yR^{-1}z \rightarrow xR^{-1}z)$



Then since R is transitive, we can draw this arrow

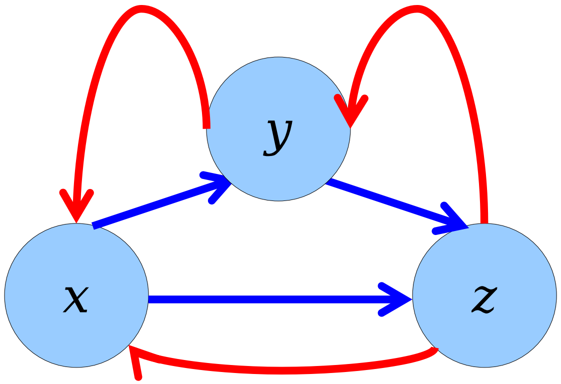


Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

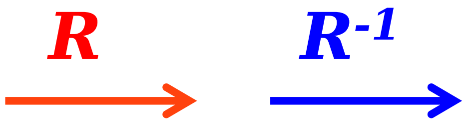
Goal:

R^{-1} is transitive

$\forall x \in A.\ \forall y \in A.\ \forall z \in A.$
$(xR^{-1}y \land yR^{-1}z \to xR^{-1}z)$

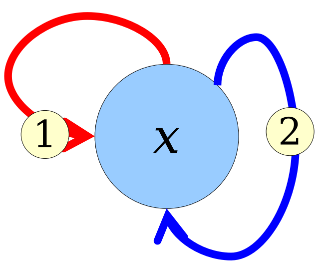


Applying the definition of R^{-1} again gives us the arrow we desire!



Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

R^{-1} is reflexive

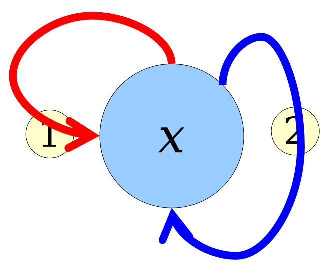


1. $x\mathbf{R}x$
(**R** is reflexive)

2. $x\mathbf{R}^{-1}x$
(definition of $\mathbf{R}^{-1}$)

Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

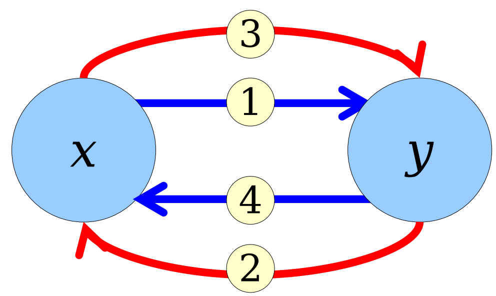
R^{-1} is reflexive



① xRx
(R is reflexive)

② $xR^{-1}x$
(definition of R^{-1})

R^{-1} is symmetric



① $xR^{-1}y$
(by assumption)

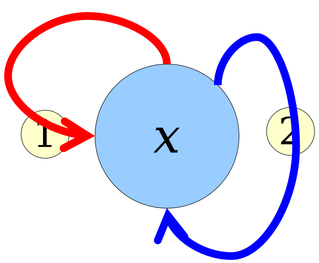
② yRx
(definition of R^{-1})

③ xRy
(R is symmetric)

④ $yR^{-1}x$
(definition of R^{-1})

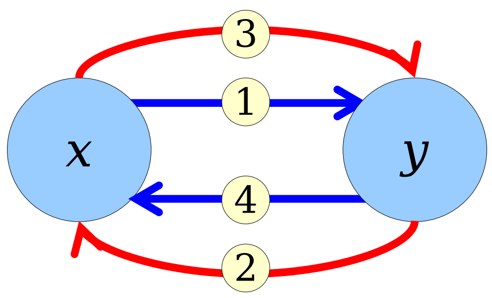
Prove or disprove: if R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

R^{-1} is reflexive



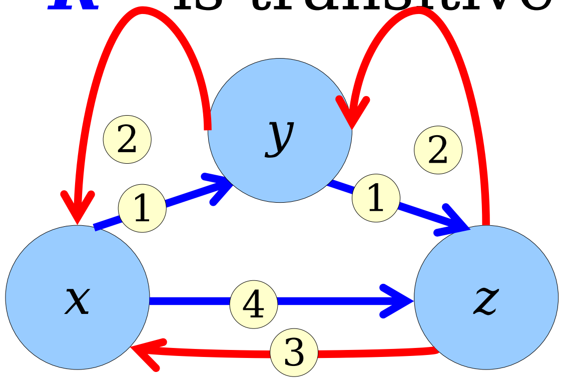
1. xRx
   (R is reflexive)
2. $xR^{-1}x$
   (definition of R^{-1})

R^{-1} is symmetric



1. $xR^{-1}y$
   (by assumption)
2. yRx
   (definition of R^{-1})
3. xRy
   (R is symmetric)
4. $yR^{-1}x$
   (definition of R^{-1})

R^{-1} is transitive



1. $xR^{-1}y$ and $yR^{-1}z$
   (by assumption)
2. yRx and zRy
   (definition of R^{-1})
3. zRx
   (R is transitive)
4. $xR^{-1}z$
   (definition of R^{-1})

Theorem: If R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Proof: Let R be an equivalence relation over a set A. We will prove that R^{-1} is also an equivalence relation over A by proving that R^{-1} is reflexive, symmetric, and transitive.

Theorem: If R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Proof: Let R be an equivalence relation over a set A. We will prove that R^{-1} is also an equivalence relation over A by proving that R^{-1} is reflexive, symmetric, and transitive.

To prove that R^{-1} is reflexive, consider any $x \in A$. We need to prove that $xR^{-1}x$. By definition, this means that we need to prove that xRx. Since R is reflexive, we know that xRx holds.

Theorem: If R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Proof: Let R be an equivalence relation over a set A. We will prove that R^{-1} is also an equivalence relation over A by proving that R^{-1} is reflexive, symmetric, and transitive.

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To prove that R^{-1} is symmetric, consider any $x, y \in A$ where $xR^{-1}y$. We need to prove that $yR^{-1}x$ holds. Since $xR^{-1}y$ holds, we know that yRx holds. Since R is symmetric and yRx is true we know that xRy is true. Therefore by definition of R^{-1}, we know that $yR^{-1}x$ holds.

Theorem: If R is an equivalence relation over A, then R^{-1} is an equivalence relation over A.

Proof: Let R be an equivalence relation over a set A. We will prove that R^{-1} is also an equivalence relation over A by proving that R^{-1} is reflexive, symmetric, and transitive.

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To prove that R^{-1} is symmetric, consider any $x, y \in A$ where $xR^{-1}y$. We need to prove that $yR^{-1}x$ holds. Since $xR^{-1}y$ holds, we know that yRx holds. Since R is symmetric and yRx is true we know that xRy is true. Therefore by definition of R^{-1}, we know that $yR^{-1}x$ holds.

Finally, to prove that R^{-1} is transitive, consider any $x, y, z \in A$ where $xR^{-1}y$ and $yR^{-1}z$. We need to prove that $xR^{-1}z$. Since $xR^{-1}y$ and $yR^{-1}z$, we know that yRx and that zRy. Since zRy and yRx, by transitivity of R we see that zRx. Thus by definition of R^{-1}, we know that $xR^{-1}z$ holds, as required. ■

Next Time

- ***Functions***
  - How do we model transformations in a mathematical sense?
- ***Domains and Codomains***
  - Type theory meets mathematics!
- ***Injections, Surjections, and Bijections***
  - Three special classes of functions.

Thought for the weekend:

Use your intuition to ask questions, not to answer them